GEOMETRY – SHEET 5 – 3×3 Orthogonal Matrices. Rotating Frames

1. Consider the orthogonal matrices

$$A = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}; \qquad B = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & -\sqrt{2} \end{pmatrix}.$$

Is either a rotation? – in which case find the axis and angle of rotation. Is either a reflection? – in which case find the plane of reflection.

2. With $0 \leqslant \theta < 2\pi$, let

$$A_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{and} \qquad B = \frac{1}{25}\begin{pmatrix} 15 & 0 & 20 \\ -16 & 15 & 12 \\ 12 & 20 & -9 \end{pmatrix}.$$

(i) Show that B is orthogonal and that $\det B = -1$. Show that B does not represent a reflection.

(ii) Find a value of θ such that $A_\theta B$ represents a reflection. For this value of θ, find the plane of reflection of $A_\theta B$.

3. Let

$$B = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & -\sqrt{2} \end{pmatrix}, \qquad R(\mathbf{i},\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad R(\mathbf{j},\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},$$

where $\theta \in \mathbb{R}$. Find α, β, γ in the ranges $-\pi < \alpha \leqslant \pi$, $0 \leqslant \beta \leqslant \pi$ and $-\pi < \gamma \leqslant \pi$ such that

$$B = R(\mathbf{i},\alpha)\, R(\mathbf{j},\beta)\, R(\mathbf{i},\gamma).$$

[Hint: note that $R(\mathbf{i},-\alpha)B\mathbf{i}$ must be a linear combination of $\mathbf{i}$ and $\mathbf{k}$.]

4. Let

$$A(t) = \frac{1}{9}\begin{pmatrix} 4+5\cos t & -4+4\cos t+3\sin t & 2-2\cos t+6\sin t \\ -4+4\cos t-3\sin t & 4+5\cos t & -2+2\cos t+6\sin t \\ 2-2\cos t-6\sin t & -2+2\cos t-6\sin t & 1+8\cos t \end{pmatrix}.$$

Given that $A(t)$ is orthogonal for all t [you do not need to verify this], find the angular velocity.

5. Let

$$\mathbf{e}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

where θ and ϕ are functions of time t.

(i) Show that $\mathbf{e}_r$ has unit length and that

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\,\mathbf{e}_\phi$$

for two unit vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ which you should determine. Find similar expressions for $\dot{\mathbf{e}}_\theta$ and $\dot{\mathbf{e}}_\phi$.

(ii) Show that $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ form a right-handed orthonormal basis.

(iii) Find the angular velocity ω, in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, such that

$$\dot{\mathbf{e}}_r = \omega \wedge \mathbf{e}_r, \qquad \dot{\mathbf{e}}_\theta = \omega \wedge \mathbf{e}_\theta, \qquad \dot{\mathbf{e}}_\phi = \omega \wedge \mathbf{e}_\phi.$$

6. (*Optional*) The matrix $A(t)$ below is orthogonal and has determinant 1. [You do not need to verify this.]

$$A(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}.$$

(i) Show that $A'(t) = WA$ where W is a constant matrix such that $W^T = -W$.

(ii) Determine W^2 and hence show that $A(t) = e^{Wt}$ where the exponential of a square matrix is defined by

$$e^X = I + X + X^2/2! + X^3/3! + \cdots.$$

(iii) Show, in general, that if X is an anti-symmetric matrix (that is $X^T = -X$) then e^X is an orthogonal matrix.